Blessing and Curse of Multivariate Return Periods

Multivariate Return Periods based on Vine Copulas

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(i) Multivariate Return Periods

Return periods describe the inter-arrival time T of extreme events usually denoting the border between the fraction of $(1 - 1/T)$ harmless and $1/T$ harmful events.

Extremes do not have a unique meaning in more than one-dimensional space as no natural ordering exists.

Design events cannot be chosen uniquely.

Copulas representing multivariate CDFs can provide multiple different critical events, resulting in *different sub-critical regions* (drawn in blue).

Kendall Return Periods, based on a copula's Kendall function, can be used to identify events defining a *critical layer*, which is defined through the cumulative distribution function as in the univariate approach identifying a *unique sub-critical region* (drawn as red hatching).

The Kendall function

$$K_C(t) = \int_{[0,1]^d} \mathbf{1}_{C(\mathbf{u}) \leq t}(\mathbf{u}) \, \mathbf{du}$$

Ensembles of possible design events might circumvent the drawback of non unique selection criteria.

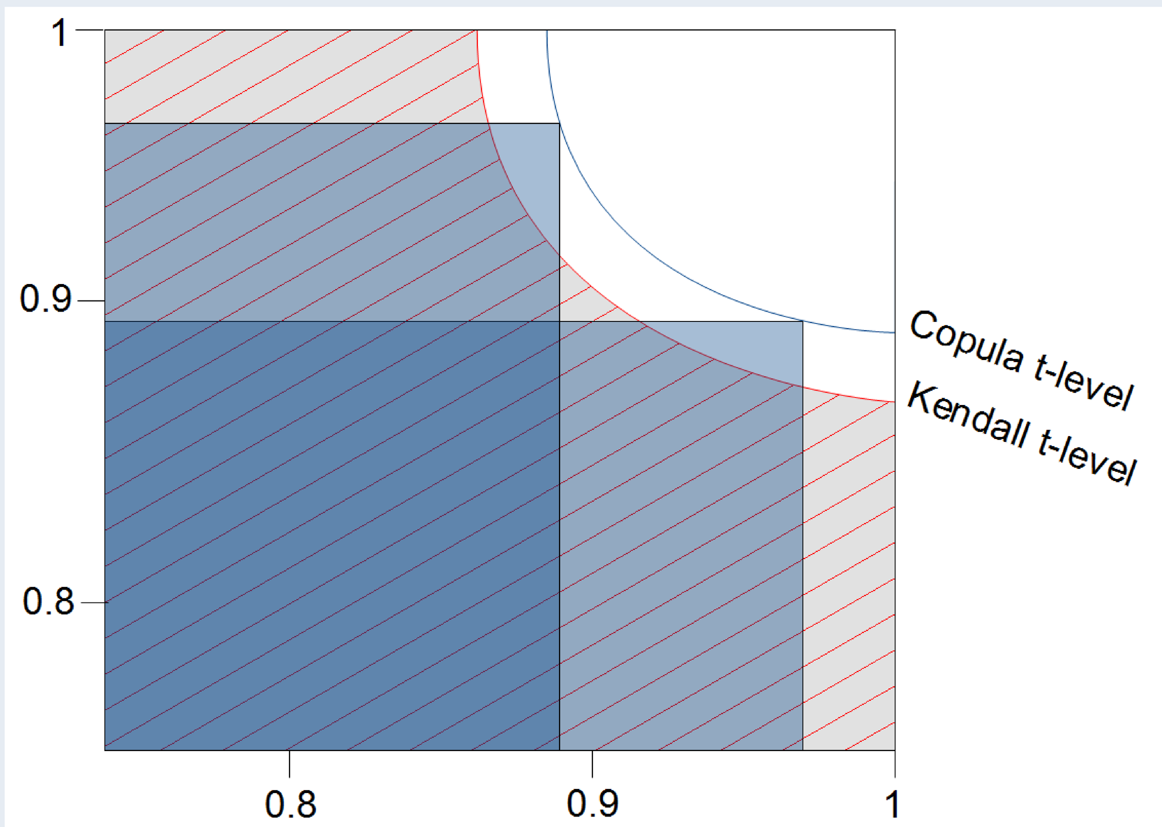


Figure: Illustration of sub-critical regions in the top right corner of the unit-square for the copula and Kendall approach. Blue rectangles indicate possible sub-critical regions for the copula approach; the red hatching indicates the unique sub-critical region of the Kendall approach. Possible design events lie along both level curves.

(ii) Vine Copulas . . .

- . . . approximate multivariate copulas, modeling *multivariate dependence structures*.
- . . . of dimension d rely on $d(d-1)/2$ bivariate copula building blocks.
- . . . allow to mix different copula families without limitations and are thus *very flexible*.
- . . . iteratively re-use well established estimation procedures for bivariate copulas.
- . . . combined with independently fitted marginal distributions yield a *full multivariate distribution* of the observed phenomenon.
- . . . allow to derive conditional densities from their multivariate density.

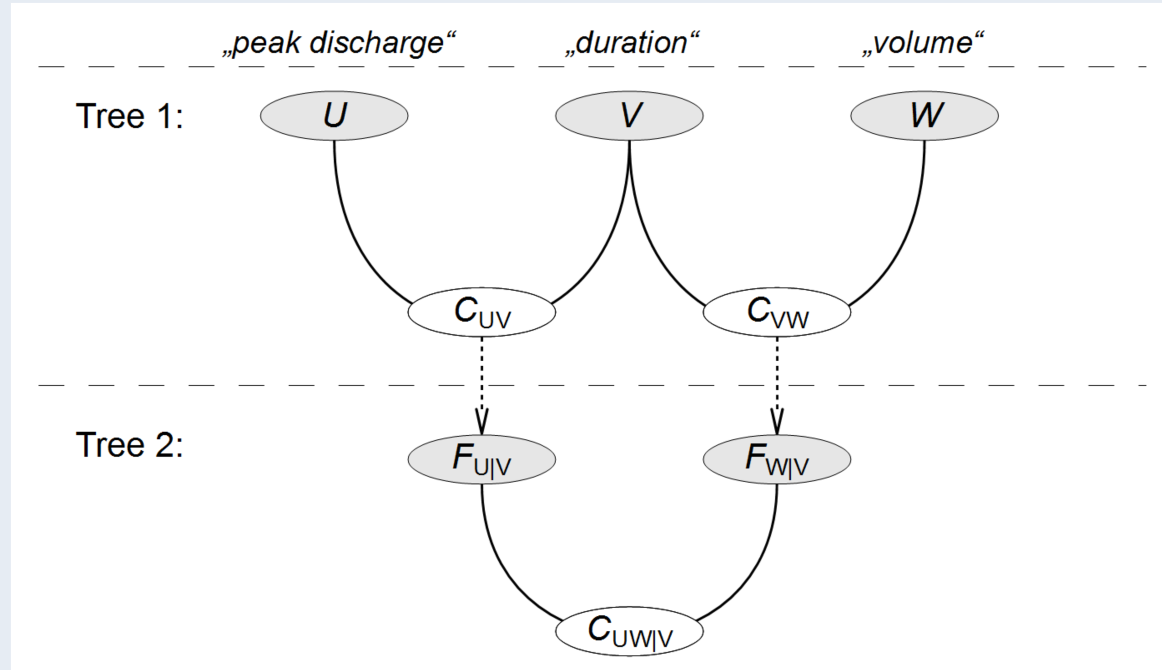


Figure: graphical representation of the 3D-vine copula

3D copula density

$$c(u,v,w) = c_{UW|V}\big(F_{U|V}(u|v), F_{W|V}(w|v)\big) \cdot c_{UV}(u,v) \cdot c_{VW}(v,w)$$

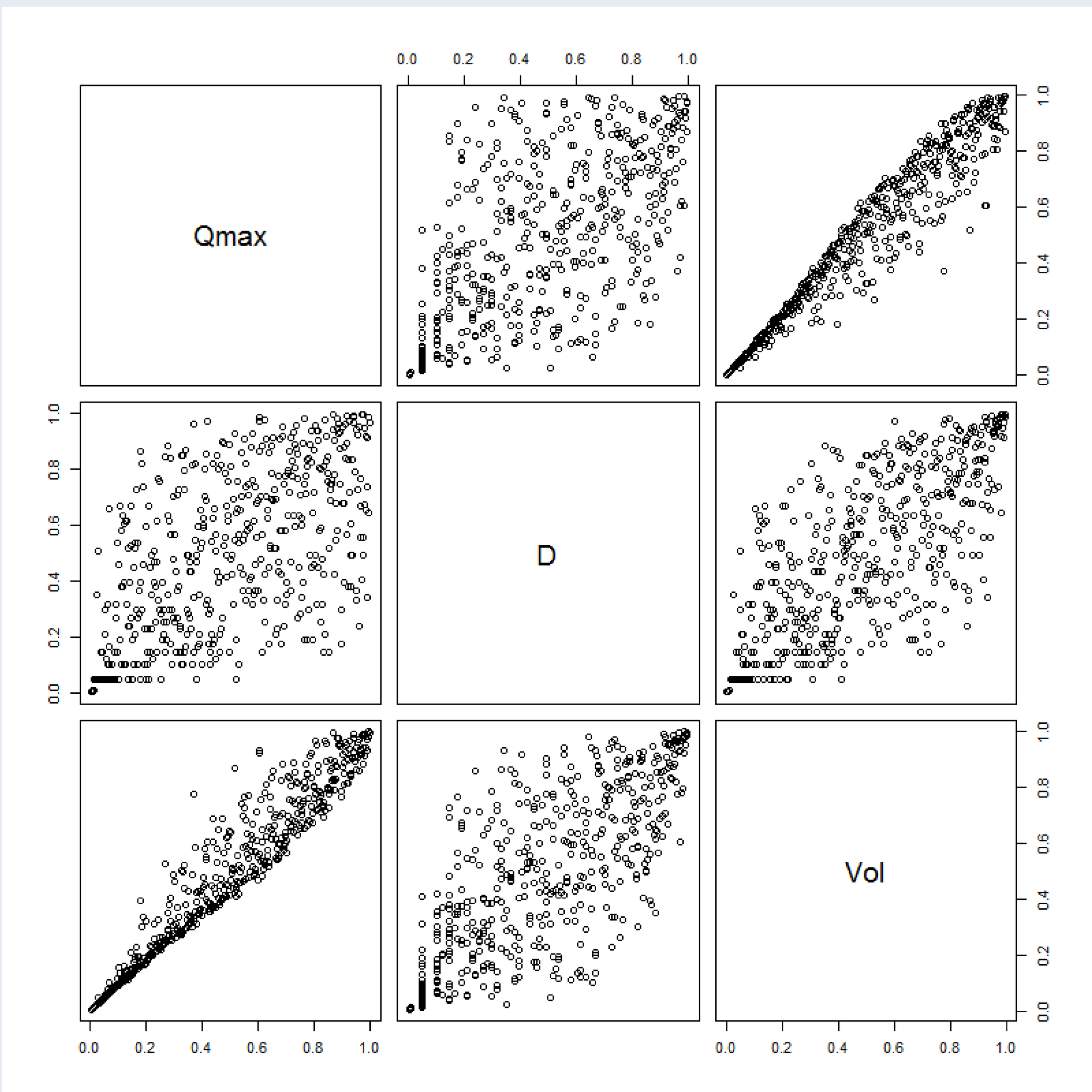
$$\text{where } F_{U|V}(u|v) = \frac{\partial C_{UV}(u,v)}{\partial v} \quad \text{and} \quad F_{W|V}(w|v) = \frac{\partial C_{VW}(v,w)}{\partial v}$$

3D full density

$$f(q_{max}, d, vol) = c\big(F_{Q_{max}}(q_{max}), F_D(d), F_{Vol}(vol)\big) \cdot f_{Q_{max}}(q_{max}) \cdot f_D(d) \cdot f_{Vol}(vol)$$

where $F_{Q_{max}}, F_D, F_{Vol}$ and $f_{Q_{max}}, f_D, f_{Vol}$ are the marginal CDFs and PDFs respectively.

(iii) Application



Data: 500 simulated triples (Q_{max}, D, Vol) of annual flood maxima at the Torbido River, central Italy

First vine tree: C_{UV}: Survival BB7 copula with parameters $\theta = 2.05$ and $\delta = 0.35$
C_{VW}: Survival BB7 copula with parameters $\theta = 2.25$ and $\delta = 1.09$

Second vine tree: $C_{UW|V}$: Student copula with parameters $\rho = 0.96$ and $\nu = 4$ degrees of freedom

Log-likelihoods: vine copula: 1047;
three-variate Gaussian copula: 935;
three-variate Archimedean copulas: 432 - 532

Three-variate return periods are derived through simulation and numerical integration.

Implementations and analysis are done in R, the developed tools can be accessed through the package *spcopula* on r-forge.

(iv) Conclusion & Outlook

CURSE

- The lack of a natural ordering involves choices affecting the results.
- Single design events from the Kendall approach might not well enough represent the sub-critical region.
- Different design events from the copula approach do not generally correspond to the same sub-critical region.

BLESSING

- Based on the log-likelihood, the vine copula is in favor of the 3-dimensional Gaussian or Archimedean copula.
- A three-variate return period improves the analysis of the phenomenon.
- The Kendall and the Copula approach can both be realized.

A comparison for a whole set of *uni- and multivariate approaches* will soon be published.
You are invited to attend the *talk Friday, 27 Apr, 15:30, Room: 34, session: HS7.5/NP8.3 (PReZI)*.

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