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Chapter 2
cokriging and indicator kriging

Seminar *Spatio-temporal dependence*,
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Outline

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Fundamentals

In (ordinary) kriging we use a linear predictor to estimate the value of a spatial random variable Z at an unknown location s_0 out of a realization $\mathbf{Z} = \big(z(s_1), \ldots, z(s_n)\big)$:

$$\hat{z}(s_0) = \sum_{i=1}^{n} \lambda_i z(s_i)$$

The weights λ_i can be derived from the following equation:

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ \mu \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_{10} \\ \vdots \\ \gamma_{n0} \\ 1 \end{pmatrix}$$

where $\gamma_{ij} = \gamma\big(Z(s_i), Z(s_j)\big) = \mathbb{E}\Big(\big(Z(s_i) - Z(s_j)\big)^2\Big)$, $0 \leq i, j \leq n$. It holds $\sum_{i=1}^{n} \lambda_i = 1$.

The idea of cokriging I

Instead of merely grounding the estimate $\hat{z}_1(s_0)$ on the observations of the variable of interest Z_1 we introduce additional variable(s) $Z_2, \ldots, Z_k$ which exhibit some correlation with the primary variable Z_1. The estimator based on a sample $\mathbf{Z} = \left(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_k\right)$ is given by:

$$\hat{z}_0(s_0) = \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda_{ij} z_j(s_i)$$

The weights are chosen to minimize the variance of the estimation error and to fulfill

$$\sum_{i=1}^{n} \lambda_{i0} = 1 \text{ and } \sum_{i=1}^{n} \lambda_{ij} = 0,\ 1 \leq j \leq k$$

The idea of cokriging II

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The weights λ_{ij} can be evaluated by

$$\begin{pmatrix} \lambda_{11} \\ \vdots \\ \lambda_{n1} \\ \lambda_{12} \\ \vdots \\ \lambda_{nk} \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{11} & \dots & \mathbf{K}_{1k} & \mathbf{1} & 0 & \dots & 0 \\ \mathbf{K}_{21} & \dots & \mathbf{K}_{2k} & 0 & \mathbf{1} & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ \mathbf{K}_{k1} & \dots & \mathbf{K}_{kk} & 0 & 0 & \dots & \mathbf{1} \\ \mathbf{1}^t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}^t & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{1}^t & 0 & 0 & \dots & 0 \end{pmatrix}^{-1} \begin{pmatrix} K_{11}(s_0, s_1) \\ \vdots \\ K_{11}(s_0, s_n) \\ K_{12}(s_0, s_1) \\ \vdots \\ K_{1k}(s_0, s_n) \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$[k(n+1) \times 1] = [k(n+1) \times k(n+1)] \cdot [k(n+1) \times 1]$$

where $\mathbf{1}^t := (1, \dots, 1) \in \mathbb{R}^n$ and each $\mathbf{K}_{ij}$ is a matrix of dimension $(n \times n)$.

The idea of cokriging III

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The matrices $\mathbf{K}_{ij}$ with $1 \leq i, j \leq k$ are defined as

$$\mathbf{K}_{ij} := \begin{pmatrix} K_{ij}(s_1, s_1) & \dots & K_{ij}(s_1, s_n) \\ \vdots & \ddots & \vdots \\ K_{ij}(s_n, s_1) & \dots & K_{ij}(s_n, s_n) \end{pmatrix}$$

while $K_{ij}(s_u, s_v)$ denotes the auto-/cross-covariance of variable Z_i at location s_u with variable Z_j at location s_v:

$$K_{ij}(s_u, s_v) := \mathrm{Cov}\big(Z_i(s_u), Z_j(s_v)\big).$$

In order to achive valid solutions, the covariance matrix needs to be positive definit.
Further details can be found in [Cressie 1993].

cokriging example - meuse

Use your own *nice* dataset or follow the meuse example.

- set up a model for primary and secondary variable
- calculate the empirical variogram
- fit a variogram
- predict

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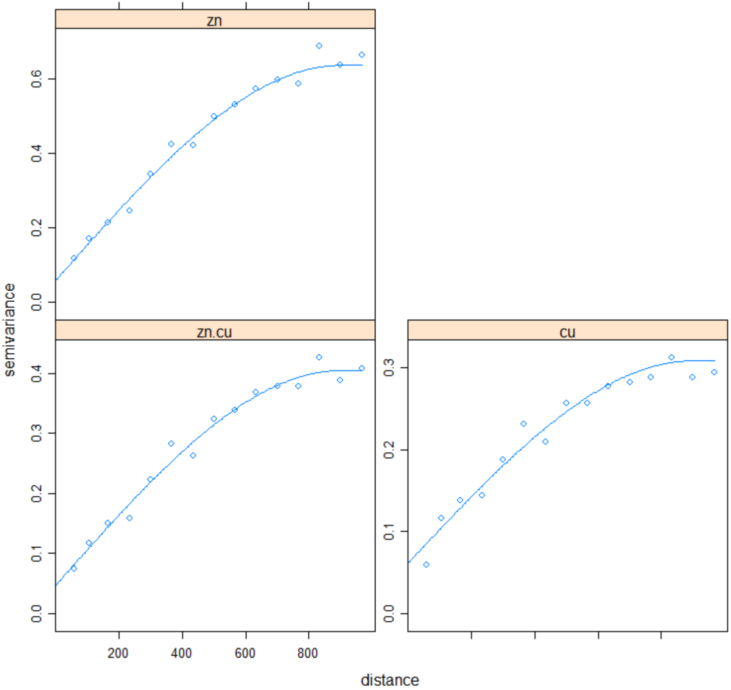
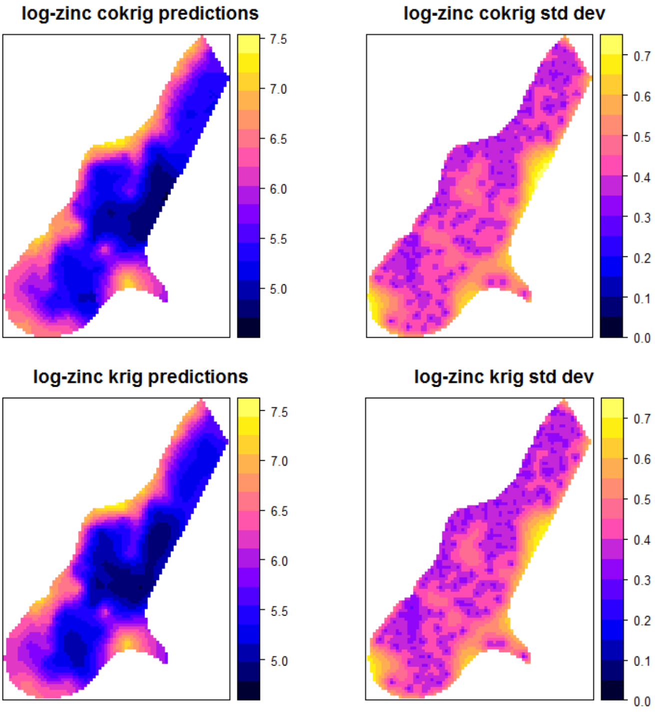


zn
zn.cu
cu
semivariance
distance
0.0
0.2
0.4
0.6
0.1
0.3
200
400
600
800



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advantage of cokriging

Cokriging might improve the interpolation if

- the primary variable is considerably undersampled
- variogram models differ in their shape

The idea of indicator kriging

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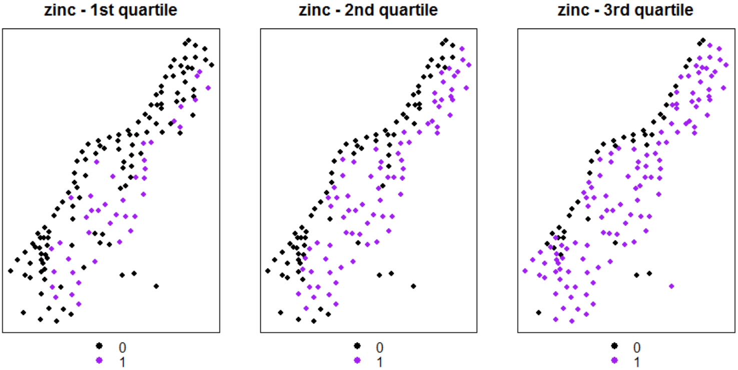
Natural processes have a varying strength of dependence for different quantiles of the distribution.
Instead of estimating a single variogram model representing a complete distribution one estimates several models for different quantiles $q \in \{q_1, \ldots, q_k\}$ described through *indicators*:

$$\mathcal{I}(Z \leq q) := \begin{cases} 1, & \text{if } Z \leq q \\ 0, & \text{if } Z > q \end{cases}$$

indicator transform

We get a new binary dataset for each quantile $z_q \in \{q_{0.25}, q_{0.50}, q_{0.75}\}$ we consider.




```
> summary(meuse[["zinc"]])
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  113.0   198.0   326.0   469.7   674.5  1839.0
```

Co-Kriging the Indicators I

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The binary datasets build up the set of co-variables. These need to be cokriged for each considered quantile $z_q \in \{q_1, \ldots, q_k\}$.

One gets *cumulative distribution function* estimates for each prediction point s_0 and each quantile z_q:

$$\hat{F}_{s_0}(z_q) = \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda_{ij} z_j(s_i)$$

The weights λ_{ij} are chosen as in the cokriging procedure.

Attention:

The estimated cumulative distribution functions do not need to be valid (i.e. exceeding $[0, 1]$ or being non-monotonic). In this case, corrections need to be applied (see [GSLIB]).

Co-Kriging the Indicators II

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Co-Kriging matrix equation loks the same, but the matrices $\mathbf{K}_{ij}$ with $1 \leq i, j \leq k$ are slightly different

$$\mathbf{K}_{ij} := \begin{pmatrix} K_{ij}(s_1, s_1) & \dots & K_{ij}(s_1, s_n) \\ \vdots & \ddots & \vdots \\ K_{ij}(s_n, s_1) & \dots & K_{ij}(s_n, s_n) \end{pmatrix}$$

where $K_{ij}(s_u, s_v)$ denotes the auto-/cross-covariance of the indicators for the i-th quantile at the location s_u and the j-th quantile at the location s_v:

$$K_{ij}(s_u, s_v) := \mathrm{Cov}\big(\mathcal{I}(Z(s_u) \leq q_i), \mathcal{I}(Z(s_v) \leq q_j)\big)$$

for a set of quantiles $\{q_1, \dots, q_k\}$.

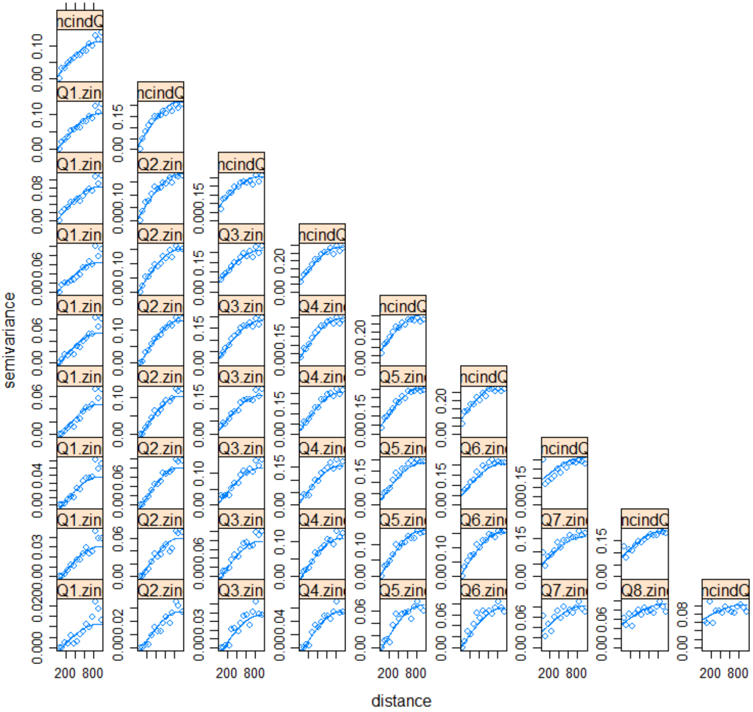
okay - but how about *real* estimates

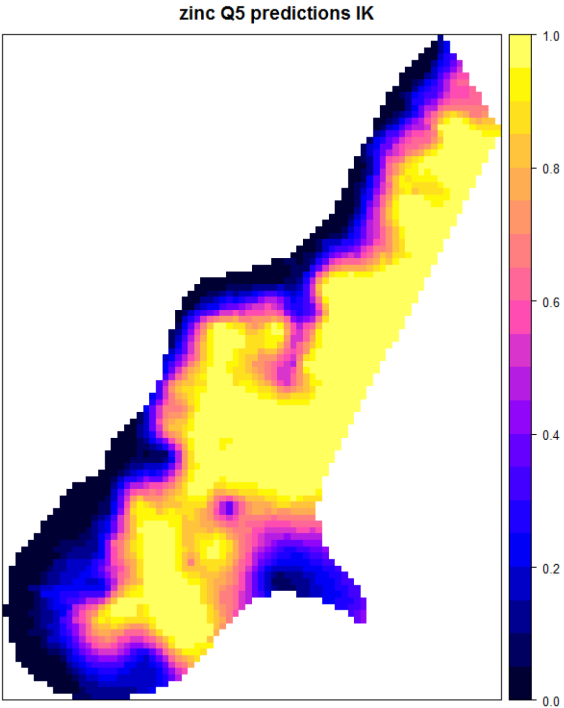
- single estimates can be derived by choosing the median of the estimated distributions at each location
- the cumulative distribution functions provide confidence intervals as well
- simulations can easily be conducted by sampling univariate random numbers and applying the inverse of the estimated cumulative distribution function

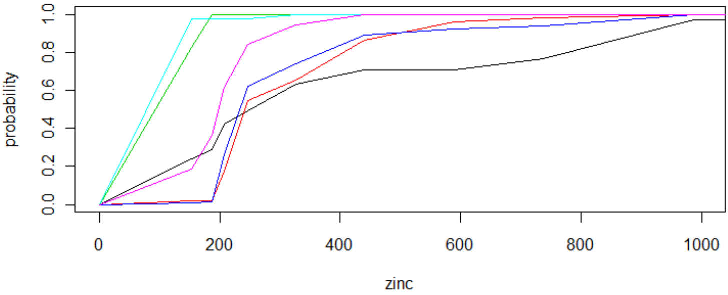
indicator kriging example - Meuse

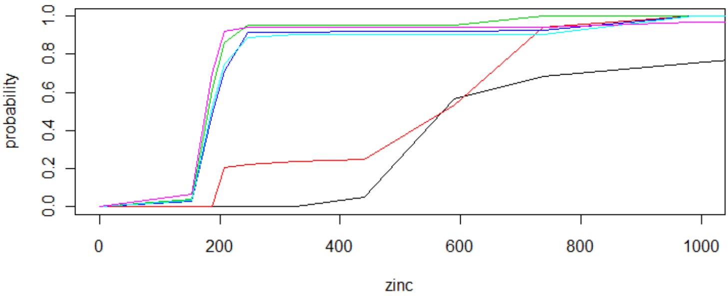
Use your own *nice* dataset or follow the Meuse example.

- break up the distribution into n+1 parts through n quantiles
- set up variograms for all combinations
- cokrig all indicators
- take a look at different cumulative distribution functions


semivariance
distance
200 800
ncindQ
Q1.zin
Q2.zin
Q3.zin
Q4.zin
Q5.zin
Q6.zin
Q7.zin
Q8.zin


zinc Q5 predictions IK
1.0
0.8
0.6
0.4
0.2
0.0


probability
0.0
0.2
0.4
0.6
0.8
1.0
0
200
400
600
800
1000
zinc


probability
0.0
0.2
0.4
0.6
0.8
1.0
0
200
400
600
800
1000
zinc

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